Laminar Shear-Stress Pattern for Nonsimilar Incompressible Boundary Layers

TSAI-TSE KAO* AND HAROLD G. ELROD JR.†
Columbia University, New York, N.Y.

A technique has been developed and applied to the solution of the laminar incompressible nonsimilar boundary-layer problem. The basic feature of this method involves the finding of a strained value of the Görtler's Principle Function such that the corresponding zero-order similarity solution provides a highly accurate approximate sheer stress result at the wall. The method can be applied locally without reference to other streamwise locations. Working charts and tables of the required universal function are provided to facilitate computations. Several examples are given to illustrate the present technique and to demonstrate its effectiveness and accuracy.

Nomenclature

C_f = local friction coefficient, $2\tau_w/\rho U_e^2$
f = dimensionless stream function
g = β-derivative of f
l = reference length
R = radius of revolution of an axisymmetric body
r = radius of cylinder or sphere
Re_x = Reynolds number, $U_e x/\nu$
U_e = local freestream velocity
U_∞ = reference velocity, also uniform freestream velocity
u = streamwise velocity component
v = transverse velocity component
x = streamwise coordinate
y = transverse coordinate normal to the surface
$\bar{\beta}$ = Görtler's Principle Function, Eq. (6)
β = strained value of $\bar{\beta}$
γ = $1/\varepsilon'$
Δ_0 = as defined in Eq. (9)
ε = $2\xi d\bar{\beta}/d\xi$
ε' = $d\varepsilon(\bar{\beta})/d\bar{\beta}$
ξ = transformed streamwise coordinate
η = transformed transverse coordinate
ν = kinematic viscosity
ρ = density
τ_w = wall shear stress
ψ = stream function

Introduction

THE purpose of this paper is to present a rapid, though accurate, method for computing wall shear-stress development in a boundary layer of incompressible fluid subjected to arbitrary freestream velocity variation. To treat this problem, methods ranging from Kármán-Polhausen procedures with relatively crude velocity profiles to full-scale numerical computation procedures have been developed. One class of methods takes as the first approximation for boundary-layer behavior the "local similarity solution—this solution being the classical Falkner-Skan Solution corresponding to local values of Görtler's Principle Function, $\bar{\beta}$. The best-known application of this local-similarity concept was that made by Lees[1] to obtain heat-transfer rates on a missile nose-cone at hypersonic speed.

Although many boundary-layer developments are well described by the local similarity model, many are not. Merk[2] was perhaps the first to propose an asymptotic expansion to account for boundary-layer nonsimilarity. The expansion takes as first approximation the local Falkner-Skan solution. Bush[3,4] later pointed out that Merk's expansion assigns incorrect magnitude to certain important terms, and Bush provided an alternative approximate solution to the problem, Dewey and Gross[5] in a survey paper, obtained another approximate solution based on Bush's method.

In a somewhat different approach, Sparrow, Quack, and Boerner[6] suggested a very interesting method to improve the local-similarity model. This model preserves some of the more attractive features of local similarity methods, while at the same time accounting for nonsimilar behavior of the boundary layer. If the shear stress information is required only at specific points, the SQB method represents a substantial saving in computer time over full-scale digital methods, but it does, nevertheless, necessitate the use of a large computer whenever shear-stress computations are to be made.

The method of Sisson,[7] as described and used by Elzy and Myers,[8] improves the local similarity approximation by using a strained value of the Principle Function as coordinate. This strained value must be found by the solution of a differential equation along the streamwise coordinate and is chosen so that a good approximation to the wall shear stress can be found from that of a shifted local similarity solution. The method is, however, subject to Bush's criticism.

The present analysis resembles, as closely as any, those of Dewey and Gross, and of Sisson. As an extension and improvement of these earlier analyses, it yields results of exceptional accuracy, even when only one correction term is applied to local similarity. Desk computations with this method are readily performed, and graphs and tables of the required universal function are provided herewith to facilitate such computation. Several examples are given to illustrate the technique and to demonstrate its effectiveness.

Transformation of Boundary-Layer Equations

The equations to be solved cover the case of incompressible axisymmetric steady flow about a body of revolutions, as shown in Fig. 1. The standard form of Prandtl's boundary-layer equations for such cases can be found in many texts. Mass continuity is satisfied automatically by introducing a stream function ψ defined by

$$u = (l/R)^j \partial\psi/\partial y \qquad v = -(l/R)^j \partial\psi/\partial x \tag{1}$$

where l is a reference length, R is the local radius of revolution of the body, u and v are velocity components along and normal

Received December 26, 1973; revision received June 3, 1974. This work is part of the doctoral dissertation of the first author.
Index category: Boundary Layers and Convective Heat Transfer—Laminar.

* Graduate Student, Department of Mechanical Engineering.

† Professor of Engineering Science, Department of Mechanical Engineering. Member AIAA.

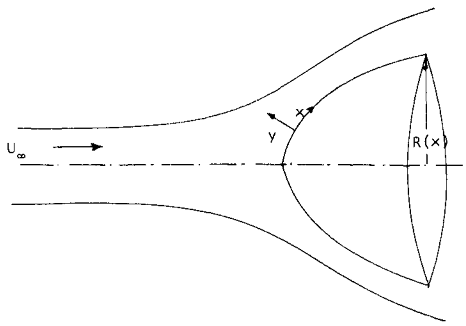

Fig. 1 Coordinate system.

to the surface of the body, $j = 1$ for axisymmetric flow, and $j = 0$ for two-dimensional flow. The velocity at the edge of the boundary layer is given either by potential flow theory or by experiment.

The (x, y) coordinates are now transformed as follows:

$$\xi = \int_0^x U_e/U_\infty (R/l)^{2j}\, dx/l$$
$$\eta = U_e (R/l)^j y/(2\nu U_\infty \xi l)^{1/2} \tag{2}$$

and a nondimensional stream function, f is also introduced

$$\psi = (2\nu l U_\infty \xi)^{1/2} f(\xi, \eta) \tag{3}$$

These transformations are due to Görtler[9] and Meskyn.[10] Their purpose is to lessen the dependence of the solution on the streamwise coordinate. The transformed momentum equation becomes

$$f''' + ff'' + \bar{\beta}(1 - f'^2) = 2\xi(f'\,\partial f'/\partial\xi - f''\,\partial f/\partial\xi) \tag{4}$$

with boundary conditions

$$f(\xi, 0) = f'(\xi, 0) = 0$$
$$f'(\xi, \infty) = 1 \tag{5}$$

Here primes denote derivatives taken with respect to η, and $\bar{\beta}$, the principal function is given by

$$\bar{\beta} = 2\xi l (l/R)^{2j} U_\infty/U_e^2\, dU_e/dx = 2(d \ln U_e/d \ln \xi) \tag{6}$$

The nondimensional shear stress, at the wall is

$$C_f (Re_x)^{1/2} = 2(R/l)^j (xU_e/2\xi l U_\infty)^{1/2} f''(\xi, 0) \tag{7}$$

where

$$C_f = 2\tau_w/\rho U_e^2$$

and

$$Re_x = U_e x/\nu$$

Expansion of Momentum Equation

In the present analysis, we now change variables from (ξ, η) coordinates to $(\bar{\beta}, \eta)$ coordinates so that the momentum equation becomes

$$f''' + ff'' + \bar{\beta}(1 - f'^2) = \varepsilon(\bar{\beta})(f'\,\partial f'/\partial\bar{\beta} - f''\,\partial f/\partial\bar{\beta}) \tag{8}$$

where

$$\varepsilon(\bar{\beta}) = 2\xi\, d\bar{\beta}/d\xi$$

If $\varepsilon(\bar{\beta})$ is zero, the preceding equation reduces to the Falkner-Skan equation with $\bar{\beta}$ as a single parameter. Merk, Bush, Dewey, and Gross suggested solving Eq. (8) by expanding f into a power series in ε to obtain improvement on the locally similar model. Here we seek instead to expand both f and $\bar{\beta}$ in a power series of ε. Thus

$$\bar{\beta} = \beta + \varepsilon(\bar{\beta})\Delta_0(\beta) + \varepsilon^2(\bar{\beta})\Delta_1 + \cdots \tag{9}$$
$$f = f_0(\beta, \eta) + \varepsilon(\bar{\beta}) f_1(\beta, \eta) + \varepsilon^2(\bar{\beta}) f_2 + \cdots \tag{10}$$

Substituting the expressions into Eq. (8) and equating like powers of ε, we find for the first two equations: Order Unity

$$f_0''' + f_0 f_0'' + \beta(1 - f_0'^2) = 0 \tag{11}$$
$$f_0(\beta, 0) = f_0'(\beta, 0) = 0, \qquad f_0'(\beta, \infty) = 1 \tag{12}$$

Order ε

$$f_1''' + f_0 f_0'' - (2\beta + \varepsilon') f_0' f_1' + (1 + \varepsilon') f_0'' f_1 =$$
$$(1 - \varepsilon'\Delta_0)(f_0'\,\partial f_0'/\partial\beta - f_0''\,\partial f_0/\partial\beta) - \Delta_0(1 - f_0'^2) \tag{13}$$

where both β and ε' appear as parameters. The function Δ_0 (and thus β) is found by specifying the following boundary conditions on Eq. (13):

$$f_1(\beta, 0) = f_1'(\beta, 0) = f_1''(\beta, 0) = 0$$
$$f_1'(\beta, \infty) = 0 \tag{14}$$

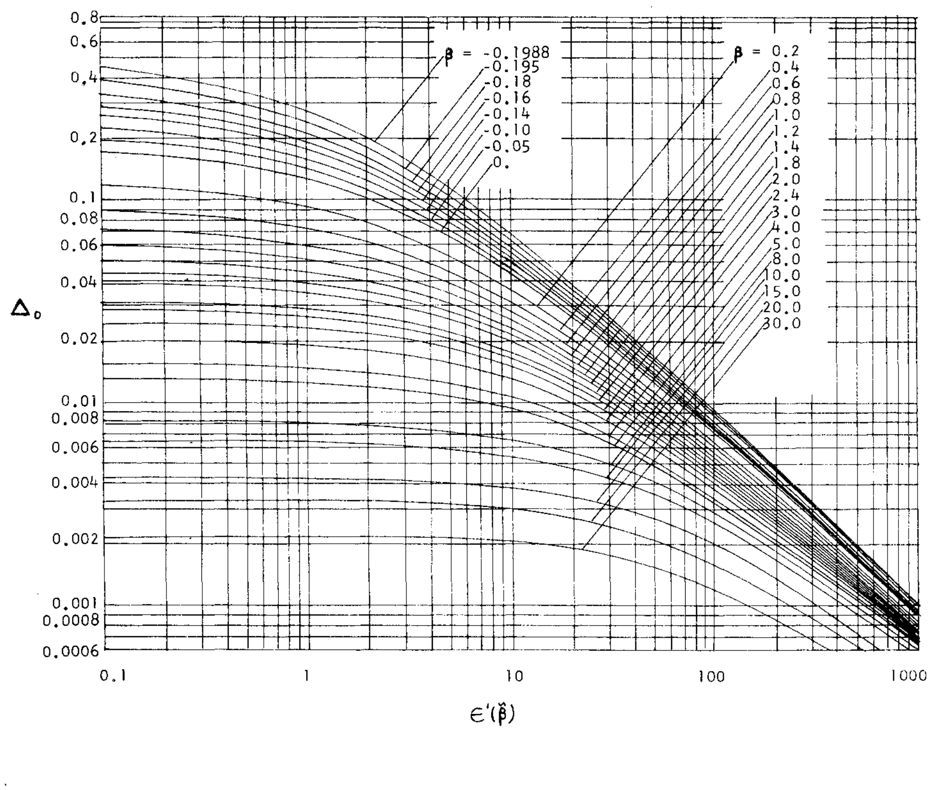

Fig. 2 Δ_0 vs $\varepsilon'(\bar{\beta})$.

Table 1 $\Delta_0(\beta, \varepsilon')$

β \ ε'	0	1	2	3	4	5	6	8	10	12	14	16
−0.1988	0.4511	0.2755	0.2017	0.1603	0.1336	0.1148	0.1008	0.0813	0.0684	0.0591	0.0520	0.0466
−0.195	0.3892	0.2442	0.1811	0.1451	0.1216	0.1049	0.0925	0.0750	0.0633	0.0548	0.0484	0.0432
−0.18	0.3314	0.2144	0.1614	0.1305	0.1101	0.0955	0.0845	0.0689	0.0543	0.0508	0.0450	0.0404
−0.16	0.2926	0.1940	0.1478	0.1205	0.1022	0.0890	0.0790	0.0648	0.0551	0.0481	0.0427	0.0384
−0.14	0.2662	0.1799	0.1384	0.1113	0.0966	0.0844	0.0752	0.0619	0.0528	0.0462	0.0411	0.0371
−0.10	0.229	0.1596	0.1246	0.1032	0.0885	0.0777	0.0696	0.0576	0.0494	0.0434	0.0387	0.0350
−0.05	0.1973	0.1415	0.1122	0.0938	0.0811	0.0716	0.0643	0.0537	0.0462	0.0407	0.0365	0.0330
0	0.1739	0.1277	0.1026	0.0865	0.0752	0.0668	0.0602	0.0505	0.0437	0.0386	0.0347	0.0315
0.20	0.1184	0.0930	0.0776	0.0672	0.0595	0.0536	0.0489	0.0418	0.0366	0.0327	0.0296	0.0271
0.40	0.0893	0.0732	0.0628	0.0553	0.0497	0.0452	0.0416	0.0361	0.0319	0.0288	0.0262	0.0241
0.60	0.0714	0.0603	0.0528	0.0472	0.0428	0.0393	0.0364	0.0319	0.0285	0.0258	0.0237	0.0219
0.80	0.0592	0.0512	0.0455	0.0411	0.0376	0.0348	0.0324	0.0287	0.0258	0.0235	0.0217	0.0201
1.00	0.0505	0.0445	0.0400	0.0365	0.0336	0.0313	0.0293	0.0261	0.0236	0.0216	0.0200	0.0186
1.20	0.0440	0.0392	0.0356	0.0327	0.0304	0.0284	0.0267	0.0240	0.0218	0.0201	0.0186	0.0174
1.40	0.0389	0.0351	0.0321	0.0297	0.0277	0.0261	0.0246	0.0222	0.0203	0.0187	0.0174	0.0163
1.80	0.0315	0.0289	−0.0268	0.0251	0.0236	0.0224	0.0212	0.0194	0.0178	0.0166	0.0155	0.0146
2.00	0.0288	0.0266	0.0248	0.0233	0.0220	0.0209	0.0199	0.0182	0.0168	0.0157	0.0147	0.0139
2.40	0.0244	0.0229	0.0215	0.0203	0.0193	0.0184	0.0175	0.0163	0.0151	0.0142	0.0134	0.0127
3.00	0.0199	0.0189	0.0179	0.0171	0.0164	0.0157	0.0151	0.0141	0.0131	0.0124	0.0118	0.0112
4.00	0.0152	0.0146	0.0140	0.0135	0.0130	0.0126	0.0122	0.0115	0.0109	0.0103	0.0099	0.0094

Thus we seek a shifted $\bar{\beta}$ which renders the wall shear stress of a particular similar solution almost locally exact.

The above analysis has one very important advantage over ordinary series expansions. If an ordinary series expansion in $\varepsilon(\bar{\beta})$ is used, as suggested by Merk,[2] Bush,[3,4] and Dewey and Gross,[5] $\bar{\beta}$ instead of β will appear in Eq. (11). As shown by Hartree[12] and later by Stewartson,[13] the solution for f_0 will be limited to the range of $\bar{\beta} \geqq -0.1988$. This will place a very severe limitation on applications, since for retarded flow, separation often occurs at values of $\bar{\beta}$ less than -0.1988. Thus the straining in $\bar{\beta}$ is crucial if a solution is to be found near separation.

The foregoing procedure resembles closely that of Sisson,[7] as outlined in Appendix C. However, there are two important differences. First, the shifted "β" is found locally—a differential equation is not required. Second, ε' can assume actual local values, whereas Sisson's procedure effectively assigns the single value $\varepsilon = 2$. These differences result in greater simplicity and improved accuracy for the present method.

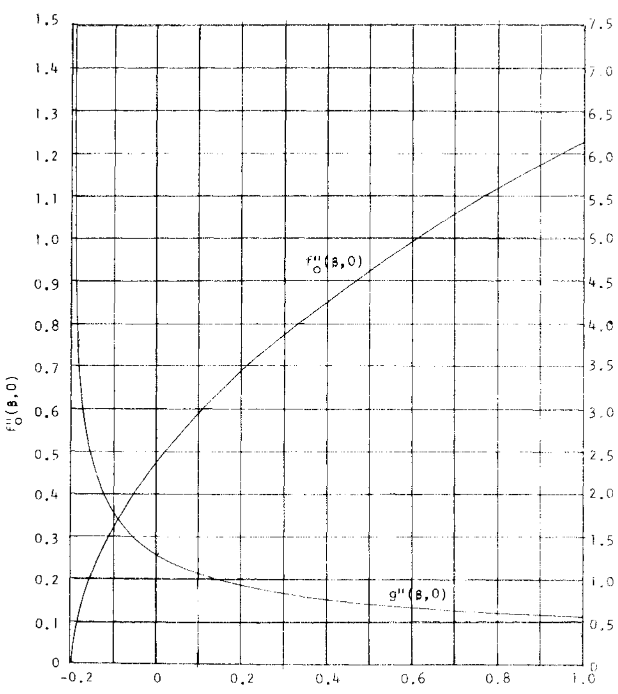

Fig. 3 $f_0''(\beta, 0)$ **and** $g''(\beta, 0)$ **vs** β.

Equation (11) is the Falkner-Skan equation, the solution of which is well known and tabulated, e.g., in Refs. 5 and 11. Equation (13) is an ordinary differential equation with homogeneous boundary conditions. However, the primary difficulty lies in computing the nonhomogeneous terms involving $\partial f_0/\partial\beta$. This is a difficult term to calculate by finite-difference method because a number of similarity solutions in the neighborhood of β must be known to a high precision. This difficulty is resolved by introducing an auxiliary equation for $g = \partial f_0/\partial\beta$ by differentiating Eq. (11) with respect to β. Thus

$$g''' + f_0 g'' - 2\beta f_0' g' + f_0'' g = -(1 - f_0'^2) \tag{15}$$

with boundary conditions

$$g(\beta, 0) = g'(\beta, 0) = g'(\beta, \infty) = 0$$

Thus Eqs. (11, 13, and 15), together with their boundary conditions form a system of ordinary differential equations, with both β and ε' appearing as parameters. A standard shooting technique is used to find the Δ_0 that satisfies all the required boundary conditions. Essential details of the method are described by Nachsheim and Swigert.[14] Numerical computations were done by us on an IBM 360/91 computer. At high values of ε' and β, values of Δ_0 were verified by means of the analyses given in Appendices A and B. The results are presented in Fig. 2 and Table 1 for various combinations of β and ε'. For convenience, the results for $f_0''(\beta, 0)$ and $g''(\beta, 0)$ are given in Fig. 3 and Table 2.

To illustrate the effectiveness of the present method, four applications will now be considered, for each of which highly accurate results have been obtained by other investigators.

The computation process for the shear stress proceeds as follows:

a) Select a streamwise location and determine from the data values for $\bar{\beta}$, $\varepsilon(\bar{\beta})$ and $\varepsilon'(\bar{\beta})$.
b) Guess β
c) Read Δ_0 from Fig. 2
d) Get a new β from Eq. (9)
e) Read a new Δ_0 from Fig. 2
f) Average the new and previous Δ_0's
g) Get a new β from Eq. (9)
h) Read Fig. 2, etc.

Table 2 $f_0''(\beta, 0)$ **and** $g''(\beta, 0)$

β	$f_0''(\beta, 0)$	$g''(\beta, 0)$
−0.1988	0	7149.0441
−0.195	0.0552	7.4603
−0.18	0.1286	3.6859
−0.16	0.1908	2.6910
−0.14	0.2397	2.2430
−0.10	0.3193	1.7856
−0.05	0.4003	1.4846
0	0.4696	1.2989
0.2	0.6867	0.9316
0.4	0.8544	0.7620
0.6	0.9958	0.6594
0.8	1.1203	0.5889
1.0	1.2326	0.5367
1.2	1.3357	0.4962
1.4	1.4316	0.4636
1.8	1.6065	0.4137
2.0	1.6872	0.3941
2.4	1.8382	0.3620
3.0	2.0439	0.3257
4.0	2.3473	0.2838

Cylinder in Crossflow

As a first example, let us consider the case of flow over a cylinder with freestream velocity given by potential theory, i.e.,

$$U_e/U_\infty = 2\sin(x/r)$$

where U_∞ is the oncoming velocity of the flow, r is the radius of the cylinder, and x is the circumferential distance measured from the forward stagnation point. With r as our reference length, the transformation according to Eqs. (2, 6, and 8) yields

$$\xi = 2[1-\cos(x/r)]$$

$$\bar{\beta} = 2\cos(x/r)/[1+\cos(x/r)]$$

$$\varepsilon(\bar{\beta}) = -4(\bar{\beta}-1)(\bar{\beta}/2-1)$$

Thus

$$\varepsilon'(\bar{\beta}) = -4(\bar{\beta}-3/2)$$

The nondimensional shear stress at wall is given by

$$C_f(Re_x)^{1/2} = f_0''(\beta, 0)\{2\sin(x/r)/[1-\cos(x/r)]\}^{1/2}$$

To demonstrate the computation procedure we select for our example a location of $x/r = 1.85$ on the cylinder. This location, being near separation, will require a maximum of iteration, and illustrate the procedure to its fullest extent as follows:

Table 3 Cylinder in crossflow

x/R	$\bar{\beta}$	ε	ε'	β	f_0''
0	1.000	0	2.000	1.000	1.233
0.927	0.750	−0.625	3.000	0.775	1.105
1.23	0.500	−1.500	4.000	0.568	0.970
1.57	0	−4.000	6.000	0.200	0.690
1.77	−0.500	−7.500	8.000	−0.080	0.352
1.81	−0.621	−8.497	8.484	−0.130	0.260
1.85	−0.761	−9.723	9.044	−0.173	0.160
1.91	−1.008	−12.078	10.031	−0.1988	0

a) $\bar{\beta} = -0.7608$; $\varepsilon(\bar{\beta}) = -9.723$; $\varepsilon'(\bar{\beta}) = 9.043$
b) Guess $\beta = -0.1988$
c) Δ_0 from Fig. 2 is 0.073
d) Eq. (9) gives $\beta = -0.0510$
e) Δ_0 from Fig. 2 is −0.049
f) Average $\Delta_0 = 0.0615$
g) New β from Eq. (9) is −0.1628
h) Δ_0 from Fig. 2 is 0.0595
i) Average $\Delta_0 = 0.0605$
j) New $\beta = -0.1725$

Further iteration yields no change in β or Δ_0. From Fig. 3, then, $f_0'' = +0.16$.

At a more typical point, say $x/r = 1.23$, the sequence converges rapidly. Thus

a) $\bar{\beta} = 0.5$; $\varepsilon(\bar{\beta}) = -1.5$; $\varepsilon'(\bar{\beta}) = 4.0$
b) $\beta = 0.5$
c) Δ_0 from Fig. 2 is 0.046
d) Eq. (9) gives new $\beta = 0.568$
e) Δ_0 from Fig. 2 is 0.045, essentially as before. Therefore, $\beta = 0.568$, and, from Fig. 3 $f_0'' = 0.97$.

The foregoing results, as well as others, are set forth in Table 3 and Fig. 4.

Comparison in Fig. 4 of our nondimensional shear stress with the finite-difference stress of Terrill[15] shows little discrepancy, even quite close to the separation point. Also shown are results from the local-similarity model, and from the two and three equation models of Sparrow's local nonsimilarity solutions.[6] A comparison of velocity distributions within the boundary layer at $x/r = 1.57$ is also presented in Fig. 5.

For airflow over a cylinder, an accurate experimental freestream velocity distribution is available from Schmidt and Wenner.[16] According to Hiemenz,[17] at $Re = 185{,}000$, this

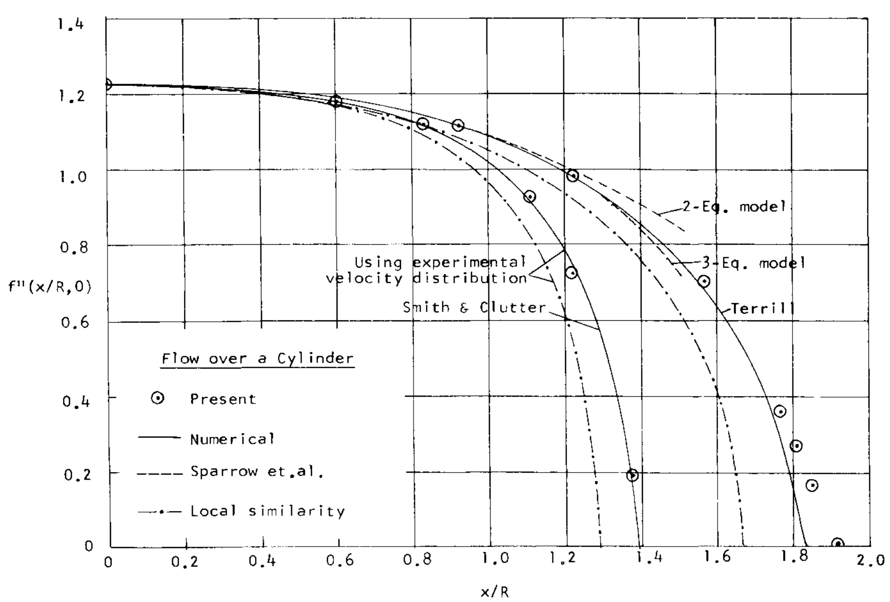

Fig. 4 Comparative predicted shear stress around a cylinder in crossflow.

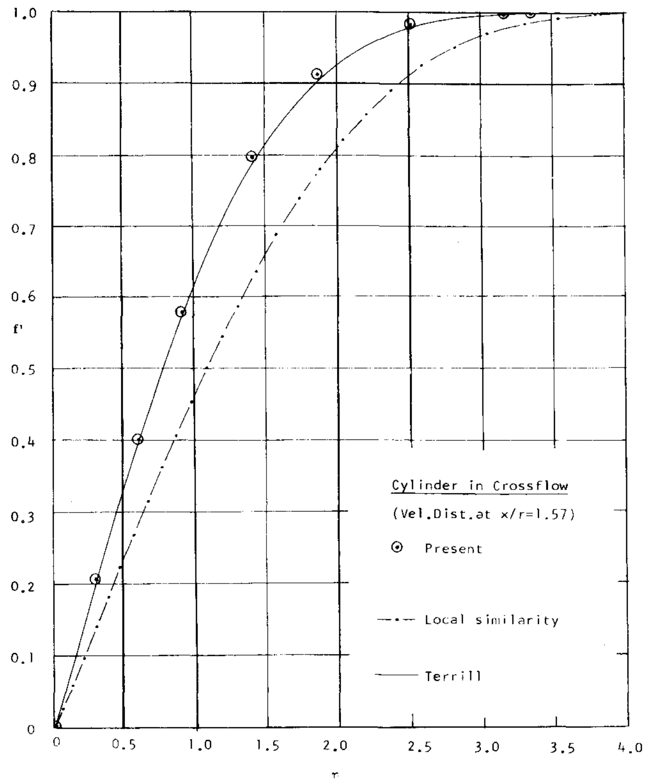

Fig. 5 Comparative predicted velocity distribution with the boundary layer.

velocity distribution can be well represented by the following empirical equation:

$$U_e(x)/U_\infty = 3.6314(x/2r) - 2.1707(x/2r)^3 - 1.5144(x/2r)^5$$

The nondimensional shear stress at several streamwise stations is shown in Fig. 4. Comparison of the present method with the accurate difference-differential method solution of Smith and Clutter[18] shows excellent agreement.

Howarth's Retarded Flow

As a more severe test, let us consider Howarth's retarded flow. This flow has been well studied by various computational procedures. The local similarity model yields a large error for this case. The freestream velocity distribution is given by

$$U_e(x)/U_\infty = 1 - x/L$$

Transformation according to Eqs. (2, 6, and 8) yields

$$\xi = x/L(1 - x/2L)$$

$$\bar{\beta} = -2x(1 - x/2L)/[L(1 - x/L)^2]$$

$$\varepsilon = -4(x/L)(1 - x/2L)/(1 - x/L)^4$$

and

$$\varepsilon'(\bar{\beta}) = 2[1 - 2x/L - (x/L)^2]/(1 - x/L)^2$$

The nondimensional shear stress at the wall is given by

$$C_f(Re_x)^{1/2} = [2(1 - x/L)/(1 - x/2L)]^{1/2} f_0''(\beta, 0)$$

Shear-stress comparisons are given in Fig. 6. In this case, the local similarity model does poorly over the entire extent of the boundary layer. On the other hand, comparison with the exact numerical solution of Smith and Clutter[18] shows that the present method is accurate even when quite close to the separation point.

Flow over a Sphere

We shall now consider the case where the nonsimilarity is due both to external freestream velocity and to curvature of the

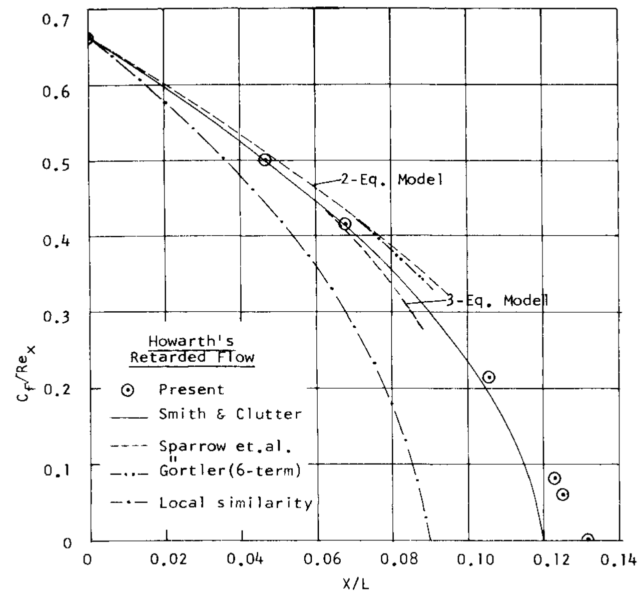

Fig. 6 Comparative predicted shear stress for Howarth's retarded flow.

body. For flow over a sphere, the freestream velocity at the edge of the boundary layer is, according to potential flow

$$U_e(x)/U_\infty = 1.5 \sin(x/r)$$

where r is the radius of the sphere. The radius of revolution, $R(x)$, is given by

$$R(x)/r = \sin(x/r)$$

The nondimensional shear stress is given by

$$C_f(Re_x)^{1/2} = 2\sin(x/r)\left[\frac{(x/r)\sin(x/r)}{2(2/3 - \cos(x/r) + \cos^3(x/r))}\right]^{1/2} f_0''(\beta, 0)$$

Results are given in Fig. 7. Here we see that the local similarity model already provides a very good approximation, but that the present method further improves the result.

Flow along a Flat Plate in Convergent Channel

As a final example, let us consider the case of accelerated flow along a plane wall in a convergent channel with a potential-flow sink existing at the point $x = L$, $y = 0$. The velocity of this accelerated outer flow is given by

$$U_e(x)/U_\infty = 1/(1 - x/L)$$

Results for f'' at several streamwise stations are given in Fig. 8. Comparison is made with Görtler's 6-term series, and the local similarity model. Our solution is in excellent agreement with Görtler's series.

Separation Criterion

In view of the good agreement achieved with other precise calculations, even quite near separation, it seems reasonable to propose a separation criterion based solely on the information required to implement the present method. Thus a curve of Δ_0 for $\beta = -0.1988$ (very near separation) is shown in Fig. 2. Equation (9) then becomes

$$\bar{\beta}_{sep} = -0.1988 + \varepsilon(\bar{\beta})\Delta_0[-0.1988, \varepsilon'(\bar{\beta})]$$

The value of ξ for which this equation is satisfied is the value for separation, and the comparisons in Figs. 4, 6–8 indicate the reliability that can be expected from predictions based on velocity distributions described by $\bar{\beta}$, $\varepsilon(\bar{\beta})$, and $\varepsilon'(\bar{\beta})$.

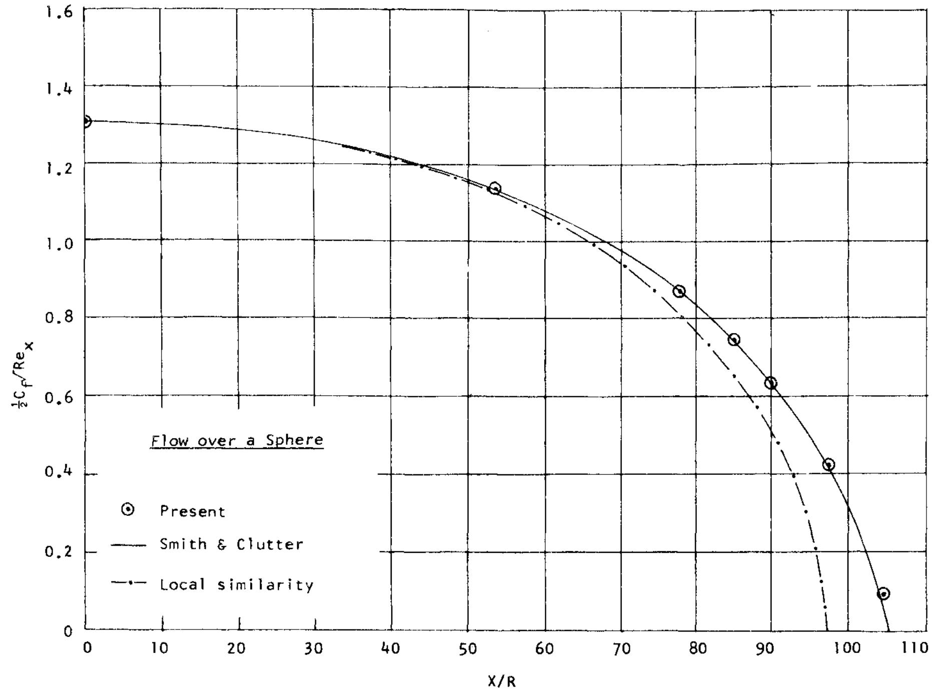

Fig. 7 Comparative predicted shear for flow over a sphere.

Conclusion

A method for determining the shear stress pattern in a laminar incompressible boundary layer developing in a flow with arbitrary pressure gradient has been described. The basic feature of the new method involves the finding of a shifted value of $\bar{\beta}$ (Görtler principal function) such that the corresponding Falkner-Skan solution provides an almost exact wall shear stress. The working graph provided (Fig. 2) makes the procedure readily implementable.

Four examples have been given to demonstrate the effectiveness and accuracy of the method. Comparisons with highly accurate alternate solutions for the examples cited indicate good agreement, even close to separation; for which a criterion is herein suggested.

Appendix A

Since the asymptotic solution of Eq. (13) for large $d\varepsilon/d\bar{\beta}$ is singular, a matched asymptotic expansion procedure is necessary. First, let us consider the outer solution. Let

$$\gamma \equiv 1/(d\varepsilon/d\bar{\beta})$$

$$f_1 = \gamma^{4/3}[\tilde{\phi}_{10} + \gamma^{1/3}\tilde{\phi}_{11} + \cdots]$$

$$\Delta_0/\gamma = \delta_0 + \gamma^{1/3}\delta_1 + \gamma^{2/3}\delta_2 + \cdots$$

Substituting these expressions into Eq. (13) and equating like powers of γ, we obtain the following set of equations:

$$(1-\delta_0)(f_0'\,\partial f_0'/\partial\beta - f_0''\,\partial f_0/\partial\beta) = 0 \quad \text{(A1)}$$

$$f_0''\tilde{\phi}_{10} - f_0'\tilde{\phi}_{10}' = -\delta_1(f_0'\,\partial f_0'/\partial\beta - f_0''\,\partial f_0/\partial\beta) \quad \text{(A2)}$$

$$f_0''\tilde{\phi}_{11} - f_0'\tilde{\phi}_{11}' = -\delta_2(f_0'\,\partial f_0'/\partial\beta - f_0''\,\partial f_0/\partial\beta) \quad \text{(A3)}$$

with outer boundary conditions given by

$$\tilde{\phi}_{10}'(\infty) = \tilde{\phi}_{1n}'(\infty) = 0; \quad n = 1, 2, \cdots \quad \text{(A4)}$$

From Eq. (A1) we conclude that

$$\delta_0 = 1 \quad \text{(A5)}$$

The solution for Eq. (A2) can be put into the following closed form. Thus

$$\tilde{\phi}_{10} = \delta_1\,\partial f_0/\partial\beta + F_0 f_0' \quad \text{(A6)}$$

where F_0 is an arbitrary constant to be found by matching with the inner solution.

The solution for Eq. (A3) takes the same form

$$\tilde{\phi}_{11} = \delta_2\,\partial f_0/\partial\beta + H_0 f_0' \quad \text{(A7)}$$

Note that the conditions on $\tilde{\phi}_{10}$ and $\tilde{\phi}_{11}$ is satisfied automatically at the outer boundary.

To find the inner solution, let the inner variable τ be chosen as follows:

$$\tau = \gamma^{-1/3}\eta$$

and let

$$f_1 = \gamma^2[\phi_{10} + \gamma^{1/3}\phi_{11} + \cdots]$$

It is also necessary to expand f_0, f_0', f_0'' $\partial f_0/\partial\beta$, and $\partial f_0'/\partial\beta$ in a power series of the inner variable. After substitution of these into Eq. (13) and equating of like powers of γ, the following set of equations results:

$$(1-\delta_0)[(ab/2)\tau^2] = 0 \quad \text{(A8)}$$

$$\phi_{10}''' - a\tau\phi_{10}' + a\phi_{10} = -\delta_0 - \delta_1(ab/2)\tau^2 \quad \text{(A9)}$$

where

$$a = f_0''(\beta, 0)$$

$$b = (\partial f_0''/\partial\beta)(\beta, 0)$$

$$\delta_0 = 1$$

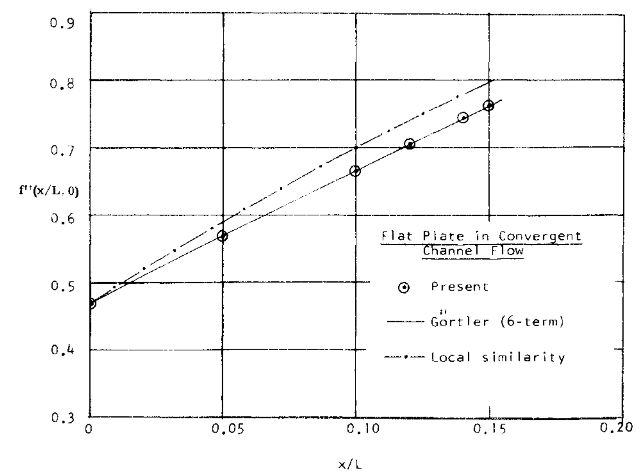

Fig. 8 Comparative predicted shear stress for a flat plate in convergent channel flow.

with boundary conditions

$$\phi_{10}(0) = \phi_{10}'(0) = \phi_{10}''(0) = 0$$

Equation (A8) requires that $\delta_0 = 1$ and is consistent with the requirement for the outer solution.

The solution for Eq. (A9) can be put into closed form. Thus:

$$\phi_{10} = A\tau \int_0^{\tau} \frac{I_{2/3}(\frac{2}{3}[ta^{1/3}]^{3/2})}{t} dt + B\tau \int_{\tau}^{\infty} \frac{K_{2/3}(\frac{2}{3}[ta^{1/3}]^{3/2})}{t} dt + \frac{\delta_1 b}{2}\tau^2 + c\tau - \frac{1}{a} \quad \text{(A10)}$$

In order to match the outer solution, A must be chosen as zero, since $I_{2/3}$ approaches infinity exponentially.

Equation (A10) can be rewritten as

$$\phi_{10} = B\left\{\alpha_0 - \tau \int_0^{\tau} \frac{\{K_{2/3}[\frac{2}{3}(ta^{1/3})^{3/2}] - \alpha_0/t\}}{t} dt + \tau \int_0^{\infty} \frac{\{K_{2/3}[\frac{2}{3}(ta^{1/3})^{3/2}] - \alpha_0/t\}}{t} dt + \frac{\delta_1 b}{2}\tau^2 + c\tau - \frac{1}{a} = B\left\{\alpha_0 - \tau \int_0^{\tau} \frac{\{K_{2/3}[\frac{2}{3}(ta^{1/3})^{3/2}] - \alpha_0/t\}}{t} dt + D\tau\right\} + \frac{\delta_1 b}{2}\tau^2 + c\tau - \frac{1}{a} \quad \text{(A11)}$$

where

$$D = \int_0^{\infty} \frac{\{K_{2/3}[\frac{2}{3}(ta^{1/3})^{3/2}] - \alpha_0/t\}}{t} dt$$

α_0 is a suitable constant, chosen in this case to be the coefficient of the singular term in the series expansion of the singular term in the series expansion of $K_{2/3}$, and is found to be equal to $0.71008a^{1/3}$. It can be shown that $D = 1.8138$.

Expanding $\phi_{10}(\tau)$ in a Taylor series about $\tau = 0$, we get

$$\phi_{10}(\tau) = B[\alpha_0 + \alpha_1\tau^2 + \cdots D\tau] + (\delta_1 b/2)\tau^2 + C\tau - 1/a \quad \text{(A12)}$$

where

$$\alpha_0 = 0.710081a^{1/3}$$
$$\alpha_1 = \frac{\pi(3)^{1/2}}{2}(0.35502)a^{1/3} \quad \text{(A13)}$$

The boundary conditions at $\tau = 0$ require that

$$B = 1/a\alpha_0 = 0.71008a^{-2/3}$$
$$C = -D/a\alpha_0 = -1.2879a^{-2/3} \quad \text{(A14)}$$
$$\delta_1 = \frac{-2B\alpha_1}{b} = \frac{-1.3717}{a^{1/3}b}$$

The complete inner solution is therefore given by

$$\phi_{10}(\tau) = 0.71008a^{-2/3}\tau \int_{\tau}^{\infty} \frac{K_{2/3}[\frac{2}{3}(ta^{1/3})^{3/2}]}{t} dt - 0.6858a^{-1/3}\tau^2 - 1.2879a^{-2/3}\tau - (1/a) \quad \text{(A15)}$$

In order to match the inner and outer solution, we define an intermediate variable as follows:

$$\rho \equiv \tau\gamma^N$$

Therefore

$$\tau = \rho\gamma^{-N}$$
$$\eta = \rho\gamma^{(1/3)-N}$$

where

$$0 < N < \tfrac{1}{3}$$

For convenience, let us choose $N = \frac{1}{6}$. The leading terms of the inner and outer solution vary, respectively, as

$$f_1^{\text{inner}} \sim C\rho\gamma^{11/6} + \frac{\delta_1 b}{2}\rho^2\gamma^{5/3} \quad \text{(A16)}$$

$$f_1^{\text{outer}} \sim F_0 a\rho\gamma^{9/6} + \left(\frac{\delta_1 b}{2} - F_0\frac{\beta}{3!}\right)\rho^2\gamma^{5/3} \quad \text{(A17)}$$

Matching requires $F_0 = 0$ with the $\gamma^{11/6}$ term to be matched by the next high-order terms from the outer solution. Therefore,

$$\tilde{\phi}_{10}(\eta) = \delta_1\, \partial f_0/\partial\beta \quad \text{(A18)}$$

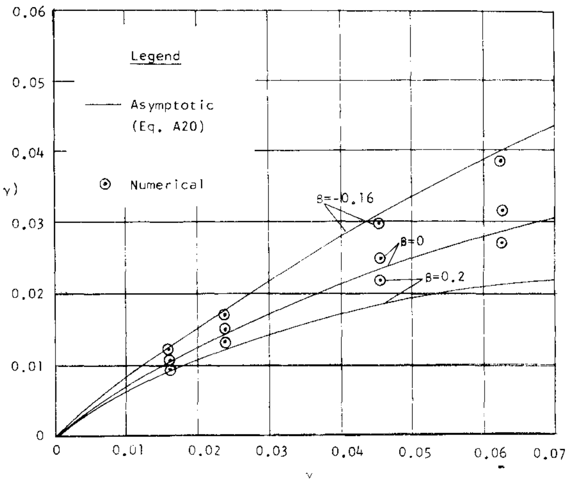

Fig. 9 Comparisons between numerical and asymptotic solutions for Δ_0 as $\varepsilon' \to \infty$.

The composite solution is given by

$$f_1(\eta) = \gamma^2\left\{0.71008a^{-2/3}\gamma^{-1/3}\eta \int_{\gamma^{-1/3}\eta}^{\infty} \frac{K_{2/3}[\frac{2}{3}(ta^{1/3})^{3/2}]}{t} dt - 1.2879a^{-2/3}\gamma^{-1/3}\eta - \frac{1}{a}\right\} - 1.3717a^{-1/3}b\gamma^{4/3}\frac{\partial f_0}{\partial\beta}(\beta,\eta) \quad \text{(A19)}$$

and

$$\Delta_0 = \gamma - \frac{1.3717}{a^{1/3}b}\gamma^{4/3} + O(\gamma^{5/3}) \quad \text{(A20)}$$

A comparison of Δ_0, obtained from Eq. (A20) and some numerical solution for various β is given in Fig. 9.

Appendix B

In order to obtain a perturbation solution for f_0, with $\beta \to \infty$, let

$$f_0 = \beta^{-1/2}F_0(\eta\beta^{1/2}) \quad \text{(B1)}$$

Substituting the preceding expression into Eq. (11), we get

$$F_0''' + (1 - F_0'^2) + (1/\beta)F_0F_0'' = 0 \quad \text{(B2)}$$

where the primes denote derivatives with respect to $(\eta\beta^{1/2})$. We seek a perturbation solution by expanding $F_0(\eta\beta^{1/2})$ in a power series of β^{-1}, thus

$$F_0(\eta\beta^{1/2}) = \mathscr{F}_{00}(\eta\beta^{1/2}) + (1/\beta)\mathscr{F}_{01} + \cdots \quad \text{(B3)}$$

Upon substituting this expression into Eq. (B2), and equating coefficients of like powers of β^{-1}, the zero-order equation is given by

$$\mathscr{F}_{00}''' + (1 - \mathscr{F}_{00}'^2) = 0$$
$$\mathscr{F}_{00}(0) = \mathscr{F}_{00}'(0) = 0; \quad \mathscr{F}_{00}'(\infty) = 1 \quad \text{(B4)}$$

The solution for Eq. (B4) had been obtained in closed form by Coles,[19] and is given by

$$\mathscr{F}_{00}(\eta\beta^{1/2}) = \eta\beta^{1/2} - 3(2)^{1/2}\chi(\eta\beta^{1/2}) + 2(3)^{1/2} \quad \text{(B5)}$$

where

$$\chi(\eta\beta^{1/2}) = \tanh\left[\frac{\eta\beta^{1/2}}{(2)^{1/2}} + \tanh^{-1}\left(\frac{2}{3}\right)^{1/2}\right] \quad \text{(B6)}$$

To obtain a perturbation solution for f_1, and thus Δ_0, let

$$f_1 = \beta^{-5/2}F_1(\eta\beta^{1/2}) \quad \text{(B7)}$$

Substituting Eq. (B7) into Eq. (13), we get

$$\frac{1}{\beta}\left[F_1''' - \left(2 + \frac{\varepsilon'}{\beta}\right)F_0'F_1' + \frac{\varepsilon'}{\beta}F_0''F_1\right] + \frac{1}{\beta^2}(F_0F_1'' + F_0''F_1) = \frac{(1 - \varepsilon'\Delta_0)}{2\beta}F_0''F_0 - \Delta_0(1 - F_0'^2) \quad \text{(B8)}$$

Table 4 δ_0 vs ε'/β

ε'/β	δ_0
0	0.06462
0.25	0.06165
0.50	0.05903
0.75	0.05668
1.00	0.05457
1.25	0.05256
1.50	0.08090
1.75	0.04930
2.00	0.04781
2.25	0.04644
2.50	0.04516
2.75	0.04396
3.00	0.04285
3.25	0.04180
3.50	0.04081
3.75	0.03988
4.00	0.03899
5.00	0.03590
6.00	0.03333
7.00	0.03117
8.00	0.02931
9.00	0.02770
10.00	0.02628

Here we treat ε'/β to be of order one, since in most cases of interest, this is true.

We now expand F_1, F_0 and Δ_0 in power series of inverse β. Thus

$$F_1 = F_{10} + \beta^{-1}F_{11} + \cdots \tag{B9}$$

$$\Delta_0 = \beta^{-1}[\delta_0 + \beta^{-1}\delta_1 + \cdots] \tag{B10}$$

Substituting Eqs. (B3, B9, and B10) into Eq. (B8), we obtain for the zero-order equation

$$F_{10}''' - \left(2 + \frac{\varepsilon'}{\beta}\right)\mathscr{F}_{00}'F_{10} + \frac{\varepsilon'}{\beta}\mathscr{F}_{00}''F_{10} = \frac{[1-(\varepsilon'/\beta)\delta_0]}{2}\mathscr{F}_{00}''\mathscr{F}_0 - \delta_0(1-\mathscr{F}_{00}'^2) \tag{B11}$$

$$F_{10}(0) = F_{10}'(0) = F_{10}'(\infty) = 0$$

Since $\mathscr{F}_{00}$ is a known function, Eq. (B11) is solved numerically on an IBM 360/91 computer, for various values of ε'/β. The solution for δ_0 is given by Table 4.

Appendix C

Since the technique used by Sisson,[7] was never published, we shall outline his method, as described by Elzy and Myers,[8] in this section. Sisson proposed to solve Eq. (4) by adopting the following expansions:

$$\tilde{\beta}(\xi) = \beta_0(\xi) + \sum_{N=1}^{\infty} a_N(\xi)\beta_N(\beta_0) \tag{C1}$$

and

$$f(\xi,\eta) = f_0(\beta_0,\eta) + \sum_{N=1}^{\infty} a_N(\xi)f_N(\beta_0,\eta) \tag{C2}$$

where

$$a_1(\xi) = 2\xi\, d\beta_0/d\xi \tag{C3}$$

However, $a_N(\xi)$ for $N \geqq 2$ are not given by the preceding authors, but they will have similar arbitrary definitions. Upon substituting the preceding expressions into Eq. (4), the following sets of ordinary differential equations resulted:

$$f_0''' + f_0 f_0'' + \beta_0(1-f_0'^2) = 0$$
$$f_0(\beta_0, 0) = f_0'(\beta_0, 0) = 0 \tag{C4}$$
$$f_0'(B_0, \infty) = 1$$

$$f_1''' + f_0 f_1'' - 2(\beta_0+1)f_0'f_1' + 3f_0''f_1 = (f_0'\,\partial f_0'/\partial\beta_0 - f_0''\,\partial f_0/\partial\beta_0) - \beta_1(1-f_0'^2) \tag{C5}$$

The function β_1 is defined by setting the boundary conditions in Eq. (C5) such that

$$f_1(\beta_0, 0) = f_1'(\beta_0, 0) = f_0''(\beta_0, 0) = 0$$
$$f_1'(\beta_0, \infty) = 0$$

It should be noted that in deriving Eq. (C5), terms of order

$$\xi \frac{d^2\beta_0}{d\xi^2} \Big/ \frac{d\beta_0}{d\xi}$$

are neglected by Sisson. Also note that in order to obtain β_0, one has to solve Eq. (C1) which is an ordinary differential equation. This often has to be solved numerically.

References

[1] Lees, L., "Laminar Heat Transfer over Blunt-Nosed Bodies at Hypersonic Flight Speed," *Jet Propulsion*, Vol. 26, No. 4, April 1956, pp. 256–268.

[2] Merk, H. J., "Rapid Calculation for Boundary Layer Transfer Solutions using Wedge and Asymptotic Expansions," *Journal of Fluid Mechanics*, Vol. 5, No. 3, April 1959, pp. 460–480.

[3] Bush, W. B., "A Method of Obtaining an Approximate Solution of Laminar Boundary Equations," *Journal of Aeronautical Science*, Vol. 28, April 1961, pp. 350–351.

[4] Bush, W. B., "Local Similarity Expansions of the Boundary-Layer Equations," *AIAA Journal*, Vol. 2, No. 10, Oct. 1964, pp. 1857–1858.

[5] Dewey, C. F., Jr. and Gross, J. F., "Exact Similar Solutions of the Laminar Boundary Layer Equations," *Advances in Heat Transfer*, Vol. 4, Academic Press, New York, 1967, pp. 317–446.

[6] Sparrow, E. M., Quack, H., and Boerner, C. J., "Local Nonsimilar Boundary-Layer Solutions," *AIAA Journal*, Vol. 8, No. 11, Nov. 1970, pp. 1936–1942.

[7] Sisson, R. M., "A New Method for the Calculation of Momentum, Heat and Mass Transfer in Laminar Boundary Layer Flow," Master's thesis, 1967, Oregon State University, Corvallis, Ore.

[8] Elzy, E. and Myers, G. A., "Engineering Calculations of Momentum, Heat and Mass Transfer through Laminar Boundaries," Bulletin 41, July 1968, Engineering Experiment Station, Oregon State University, Corvallis, Ore.

[9] Görtler, H., "A New Series for the Calculation of Steady Laminar Boundary Layer Flows," *Journal of Mathematics and Mechanics*, Vol. 6, No. 1, Jan. 1957, pp. 1–66.

[10] Meksyn, D., *New Methods in Laminar Boundary Layer Theory*, Pergamon Press, Oxford, 1961.

[11] Elzy, E. and Sisson, R. M., "Tables of Similar Solutions to the Equations of Momentum, Heat and Mass Transfer in Laminar Boundary Layers of Flow," Bulletin 40, 1967, Engineering Experiment Station, Oregon State University, Corvallis, Ore.

[12] Hartree, D. R., "On an Equation Occurring in Falkner-Skan Approximate Treatment of the Equations of the Boundary-layers," *Proceedings of Cambridge Philosophical Society*, Vol. 33, 1937, pp. 223–239.

[13] Stewartson, K., "Further Solutions of Falkner and Skan Equation," *Proceedings of Cambridge Philosophical Society*, Vol. 50, 1954, pp. 454–465.

[14] Nachsheim, P. R. and Swigert, P., "Satisfaction of Asymptotic Boundary Conditions in Numerical Solution of Systems of Nonlinear Equations of the Boundary Layer Type," TN D 3004, 1965, NASA.

[15] Terrill, R. M., "Laminar Boundary Layer Flow Near Separation with and without Suction," *Philosophical Transaction of the Royal Society of London*, Ser. A, Vol. 253, No. 1022, Sept. 1960, pp. 55–100.

[16] Schmidt, E. and Wenner, K., "Wärmeabgabe über der Umfand eines angeblasenen geheizten zylinders," *Forsch, Ing.-Wes.*, Vol. 12, 1941; English translation, TM 1050, 1943, NACA.

[17] Hiemenz, K., "Die Grenzschicht an einem in den gleichformigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder," *Dingl. Polytech. Journal*, Vol. 326, 1911, pp. 321–342.

[18] Smith, A. M. O. and Clutter, D. W., "Solutions of the Incompressible Laminar Boundary-Layer Equations," *AIAA Journal*, Vol. 1, No. 9, Sept. 1963, pp. 2062, 2071.

[19] Coles, D. E., "The Laminar Boundary Layer Near a Sonic Throat," *Proceedings of Heat Transfer and Fluid Mechanics Institute*, Stanford University Press, Stanford, Calif., 1957, pp. 118–137.